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On the Hardening Response in Small Deformation of Metals

by

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Abstract. This paper is concerned with a special class of hardening response functions for small deformation of elastic-plastic materials, its application to isotropic metals, and comparison of the theoretical results with experimental cyclic stress-strain curves for two different metals. The theoretical development is carried out within the scope of the existing purely mechanical theory for the "rate-independent" response of elastic-plastic materials, which admits the existence of a single loading function, as well as certain accepted idealizations. After summarizing the basic equations for small deformation, detailed attention is given to the development of a special form of the hardening response function, motivated mainly by the observation that the stress-strain curves for uniaxial cyclic loading of a fairly large class of metals attain -- after several cycles -- the so-called saturation hardening. We exploit this property; and, in the case of isotropic metals, systematically derive some restrictions on the constitutive coefficients in the loading function and the hardening response. Comparison of the results with two sets of experimental data, obtained from uniaxial cyclic loading of a 304 stainless steel and a 2024 aluminum alloy, shows good agreement within the understood idealizations of the basic theory.

1. Introduction

By way of background, it may be recalled that in the usual development of the general theory of elastic-plastic materials, the rate of plastic strain and the rate of work-hardening are expressed as linear functions of either the rate of stress or the rate of strain. However, the coefficient response functions in these equations, as well as the loading function and other constitutive response functions, are all independent of the rate of stress, the rate of strain and time derivatives of other kinematical ingredients. For this reason, such a theory is sometimes called a "rate-independent" theory of plasticity, in contrast to a theory which characterizes "rate-dependent" behavior by including rate quantities in the loading function and/or other constitutive response functions. Moreover, since the "rate-independent" formulation of plasticity theory employs constitutive equations for the rate of plastic strain and rate of work-hardening, it is also referred to as a rate-type theory in contradistinction to a functional-type theory in which the material response depends on the entire past history of deformation.

The development of the present paper is based on the rate-type theory of elastic-plastic materials given by Green and Naghdi [1,2], which is constructed relative to loading surfaces in stress space.† We also utilize the corresponding formulation of the rate-type theory given by Naghdi and Trapp [3] in which the basic constitutive equations are expressed relative to loading surfaces in strain space.* Our aim is to find an acceptable description for the hardening

†The theory of elastic-plastic materials in [1,2] includes thermal effects and is developed within the framework of a thermodynamical theory, while the development in [3] is carried out in the context of the mechanical theory. Although we confine attention to the purely mechanical aspects of the subject, the basic theory utilized here can easily be interpreted in the context of the isothermal theory. The purely mechanical theory of elastic-plastic materials with large deformation corresponds to the general developments in [1] prior to thermodynamical restrictions. We employ here a second form of the theory of plasticity relative to loading surface in stress space given in Section 4 of [2].

*Also, reference may be made to Naghdi [4] which includes a brief summary of the main results in [1,2,3], as well as a discussion of some advantages of the formulation relative to loading surfaces in strain space.

response in small deformation of metals within the scope of the rate-type theory of plasticity.

Elastic-plastic strain hardening in metals has received considerable attention in the recent literature. General dissatisfaction has been expressed with the more traditional kinematic and isotropic hardening rules, especially for describing the behavior of metals under cyclic loading. Rather than seek a more general representation of the hardening response within the scope of the existing rate-type theory of plasticity, many recent studies have sought instead to introduce modifications in the basic theory. Before discussing the nature of these modifications, it is useful to review briefly certain idealizations which are incorporated in the existing rate-type theory of elastic-plastic materials when the deformation is infinitesimal. General background information concerning these idealizations can be found in the papers of Drucker [5] and Naghdi [6].

Consider the uniaxial stress-strain curves shown in Fig. 1(a) and 1(b). Such schematic behavior is typical of ductile metals in a homogeneous deformation. Figure 1(a) shows loading in tension followed by unloading to zero stress, followed by reloading in tension. Figure 1(b) is similar, except that reloading occurs in compression. In the construction of the basic theory of plasticity, it is customary to idealize the reloading segment FEG by the dashed line FEDG. The loading program in Fig. 1(b) differs from that in Fig. 1(a) only in that reloading occurs in compression. Beyond this fact, the two schematic figures are similar, and the actual material response bears this out: segment LN is similar to EG. It also seems consistent, therefore, to idealize KLN by KLMN. In the same spirit, segments AC and HJ are idealized by ABC and HIJ, respectively.

Some of the idealizations noted above are incorporated in the rate-type theory of plasticity through the concept of a <u>yield</u> or a <u>loading</u> surface, which places a sharp distinction between elastic and plastic deformation. If the yield surface

is introduced in the usual way, then a rate-type theory of plasticity with small deformation should not be expected to describe the smooth transition regions such as EG and LN in Fig. 1(a) and 1(b). Recent investigations have tried to better represent these transition regions by either introducing additional yield surfaces [7,8,9,10] or by eliminating the yield surface entirely.‡ Eisenberg [12] retains the idealization EDG (see Fig. 1(a)), but chooses to describe the segments AC and LN more precisely by introducing a discrete memory of the so-called "last significant loading event."

At best, however, the transition regions described above are second order to the main features of the material response; and it seems premature to introduce refinements that are meant to improve the description of these regions before the main features of the material response have been represented reasonably well in the context of the basic theory. Certainly the shortcomings of the isotropic and kinematic hardening rules are not entirely due to idealizations of the rate-type theory. By the same token, a detailed account of the transition regions should not be looked upon as a remedy for poor description of the main features of the material behavior.

After a brief review of the basic constitutive equations for an elastic-plastic material in section 2, we examine in section 3 a linearized version of the theory appropriate for small deformation of metals. General observations pertaining to metal hardening, including saturation hardening, are used to motivate restrictions and special assumptions on the form of the response

‡In recent years some authors appear to have placed special emphasis either on the possibility of developing a theory of plasticity without introducing a 'yield surface' or on requiring that the theory be capable of predicting the existence of a 'yield surface' rather than postulating the existence of a yield or loading surface <u>ab initio</u>. As has been pointed out already by Green and Naghdi [11], such endeavors appear at present to be somewhat illusory in the following sense: whichever way the theory is developed, it must necessarily involve some assumptions which ultimately result in a yield surface; and it appears to be largely a matter of taste as to which kind of assumptions are preferred at the outset of the development of the theory. Additional remarks and some related references on this point can be found in [11].

functions. Isotropic metals are considered in section 4 and the number of independent material coefficients describing the hardening response is reduced to six. These results include, of course, the isotropic and kinematic hardening rules as special cases. In section 5, the reduced constitutive equations are expressed relative to a loading surface in strain space and restrictions on the material coefficients are derived. These restrictions are less stringent than those which can be deduced via a corresponding development relative to stress space. A procedure for determining the six material constants from uniaxial cyclic loading data is outlined in section[§] 6. In section 7, the theoretical predictions of material response are compared to some new and existing experimental data in uniaxial cyclic loading. First, the cyclic stress-strain data for 304 stainless steel obtained by Pugh _et al_. [13] are compared to predicted stress-strain curves based on the present developments. Next comparison is made with experiments on 2024 aluminum alloy, which were conducted at Berkeley by the present authors. In these experiments, cylindrical specimens (of solid cross-section) were cycled in tension-compression through two different loading programs of two strain cycles each. For both the stainless steel and the aluminum alloy, comparison of the theory and experiment shows good agreement within the understood idealizations of the basic theory.

[§] Details of the calculations for the constitutive restrictions in section 5 and for the determination of the material constants in section 6 are provided, respectively, in Appendices A and B at the end of the paper.

2. Notation and Basic Equations.

Let the motion of a body be referred to a fixed system of rectangular Cartesian coordinate axes and let a typical particle of the body occupy the position X_A in some fixed reference configuration. Further, let x_i designate the position of the particle in the present configuration at time t. Then, a motion of the body is defined by

$$x_i = \chi_i(X_K,t) \tag{1}$$

and the deformation gradient relative to the reference position is denoted by

$$F_{iK} = \frac{\partial \chi_i}{\partial X_K} \ . \tag{2}$$

Throughout the paper, indices take the values 1,2,3 and the usual summation convention over repeated indices is employed. Lower case indices are associated with the spatial coordinates x_i and upper case indices refer to the material coordinates X_A. The Lagrangian strain tensor e_{AB} is defined by

$$e_{KL} = \tfrac{1}{2}(c_{KL} - \delta_{KL}) \quad , \quad c_{KL} = F_{iK}F_{iL} \quad , \tag{3}$$

where c_{KL} is the Cauchy-Green measure of deformation and δ_{KL} is the Kronecker delta.

We summarize here the main constitutive equations of the purely mechanical theory of elastic-plastic materials contained in the work of Green and Naghdi [1,2]. In addition to the strain tensor e_{AB}, at each material point we admit the existence of (i) a plastic strain specified by a symmetric second-order tensor† $e^p_{KL} = e^p_{KL}(X_M,t)$ which has the same invariance properties as e_{KL}; (ii) a scalar-valued function $\kappa = \kappa(X_M,t)$ as a measure of work-hardening; and (iii) a scalar-valued yield (or loading) function $f(s_{KL},e^p_{MN},\kappa)$, where s_{KL} is the

†As shown by Naghdi and Trapp [14], symmetry of the plastic strain follows as one result of a work inequality postulated over a closed cycle of homogeneous deformation in strain space.

symmetric Piola-Kirchhoff stress tensor. For fixed values of e^p_{MN} and κ, the equation

$$f(s_{KL}, e^p_{MN}, \kappa) = 0 \tag{4}$$

defines a closed orientable surface of dimension five in the six-dimensional Euclidean space of the symmetric stress s_{KL}. We assume the loading function f to be continuously differentiable with respect to its arguments.

The constitutive equations for the rate of plastic strain and the rate of work-hardening, relative to the loading function f, can be expressed in the form

$$\dot{e}^p_{KL} = \begin{cases} 0 & \text{when } f < 0 \ , \\ 0 & \text{when } f = 0 \text{ and } \hat{f} < 0 \ , \\ 0 & \text{when } f = 0 \text{ and } \hat{f} = 0 \ , \\ \lambda \beta_{KL} \hat{f} & \text{when } f = 0 \text{ and } \hat{f} > 0 \ , \end{cases} \tag{5}$$

and

$$\dot{\kappa} = h_{KL} \dot{e}^p_{KL} \ , \tag{6}$$

where a superposed dot stands for the material time derivative holding X_A fixed and where $\hat{f}$ is defined by

$$\hat{f} = \frac{\partial f}{\partial s_{KL}} \dot{s}_{KL} \ . \tag{7}$$

Also, the scalar λ is restricted to assume positive values and h_{KL} and β_{KL} are symmetric second order tensors. Each of the latter three quantities is a function of the list of variables

$$\mathcal{V} = (s_{MN}, e^p_{MN}, \kappa) \ . \tag{8}$$

The four conditions involving f and $\hat{f}$ on the right-hand side of (5) are called

the loading criteria. Using the conventional terminology, in the order listed they are: the state of elastic deformation, unloading from an elastic-plastic state, neutral loading and loading from an elastic-plastic state.

In the purely mechanical theory, it remains to introduce the constitutive equation for the stress and this is specified by

$$s_{KL} = \hat{s}_{KL}(\mathfrak{u}) \quad , \quad \mathfrak{u} = (e_{MN}, e^p_{MN}, \kappa) \quad , \tag{9}$$

with the added restriction that

$$\frac{\partial \hat{s}_{KL}}{\partial e_{MN}} = \frac{\partial \hat{s}_{MN}}{\partial e_{KL}} \quad . \tag{10}$$

We note that the constitutive equation $(9)_1$ holds during loading, as well as during unloading and neutral loading, and that the condition (10) ensures that the stress is derivable from a potential. We assume that for fixed values of e^p_{MN} and κ the expression $(9)_1$ is invertible in the form

$$e_{KL} = \hat{e}_{KL}(\mathfrak{v}) \quad . \tag{11}$$

The preceding summary of the basic equations for an elastic-plastic material began with equation (4) and hence is based on the concept of a yield surface and loading criteria in stress space. Following Naghdi and Trapp [3], it is possible to provide an equivalent formulation of the basic equations relative to loading surfaces in strain space and this has certain advantages which are discussed in [3]. However, since a formulation relative to loading surfaces in stress space is useful for certain interpretations, we use this formulation in sections 3 and 4 to motivate special assumptions on the hardening response function h_{KL} and the loading function f. In section 5, the results of sections 3 and 4 are recast in their corresponding form relative to loading surfaces in strain space because in that form the derived restrictions on the constitutive coefficients are less stringent.

Although the above constitutive equations are fully general in the context of the purely mechanical theory, it is possible to conceive of a plausible physical hypothesis that will place restrictions on their form. Starting with an assumption for the work done by body forces and surface tractions in a closed cycle of homogeneous deformation, Naghdi and Trapp [14,15] have shown, among other results, that during <u>loading</u> the plastic strain rate must satisfy

$$\left(\frac{\partial \hat{s}_{KL}}{\partial e^p_{MN}} + \frac{\partial \hat{s}_{KL}}{\partial \kappa} h_{MN}\right)\dot{e}^p_{MN} = -\gamma \frac{\partial f}{\partial s_{MN}} \frac{\partial \hat{s}_{MN}}{\partial e_{KL}} \quad , \quad \gamma \geqq 0 \quad , \tag{12}$$

where γ is a nonnegative scalar function of the variables (8) and the stress rate. Also recall that by expressing the stress response in the alternative form (see [14,16])

$$s_{KL} = \bar{s}_{KL}(e_{MN}-e^p_{MN}, e^p_{MN}, \kappa) \quad , \tag{13}$$

the relation (12) may be rewritten as

$$\dot{e}^p_{KL} - \frac{\partial \hat{e}_{KL}}{\partial s_{PQ}} \left(\frac{\partial \bar{s}_{PQ}}{\partial e^p_{MN}} + \frac{\partial \bar{s}_{PQ}}{\partial \kappa} h_{MN}\right)\dot{e}^p_{MN} = \gamma \frac{\partial f}{\partial s_{KL}} \quad . \tag{14}$$

It is clear from (14) that if the stress response in (13) depends only on $e_{MN} - e^p_{MN}$ so that $\partial \bar{s}_{KL}/\partial e^p_{MN} = 0$ and $\partial \bar{s}_{KL}/\partial \kappa = 0$, then the plastic strain rate is directed along the normal to the loading surface in stress space.

In the remainder of this paper we confine attention to a discussion of the elastic-plastic constitutive equations for infinitesimal strain. Hence, we assume that the displacement, the displacement gradient, the plastic strain and the stress (when expressed in a suitable nondimensional form), as well as their space and time derivatives, are all small of order ϵ. Consistent with these, the loading function f, the work-hardening parameter κ and its time derivative $\dot{\kappa}$ must be at least of order ϵ^2 (see the discussion in section 3). In terms of the components of the displacement vector u_A, the infinitesimal strain tensor e_{KL} is defined by

$$u_A = x_i \delta_{iA} - X_A = O(\epsilon) \quad , \quad e_{KL} = \tfrac{1}{2}(u_{K,L} + u_{L,K}) + O(\epsilon^2) = O(\epsilon) \quad , \tag{15}$$

where a comma stands for partial differentiation with respect to the material coordinates X_A, which to the order ϵ are coincident with x_i. The infinitesimal elastic strain tensor e^e_{KL} is defined by

$$e^e_{KL} = e_{KL} - e^p_{KL} \quad . \tag{16}$$

We also recall here that in the linearized version of the theory the distinction among the various representations for the stress, namely the nonsymmetric Piola-Kirchhoff, the symmetric Piola-Kirchhoff and the Cauchy stress, disappears.

In the next section we discuss a form of the preceding constitutive equations appropriate for metallic materials undergoing small deformations.

3. Work-hardening of Metals in Small Deformation.

Because many ideas in plasticity theory are generalizations of observations made of material behavior in one dimension, it is useful to first examine the response of a typical metal under a program of uniaxial cyclic loading between constant strain limits. Consider the loading process illustrated by the stress-strain curve in Fig. 2. Loading begins in tension from zero strain (point O) on a sample in the virgin state. As long as the stress remains in the initial elastic range, the stress response is observed to be linear in the strain. After the initial yield point (labeled point A) is reached, the plastic strain increases continuously from zero until unloading takes place at point B. During unloading from B, the stress response is again linear in the strain and provided that the strain at B is small, the line BC is usually observed to be parallel to the stress-strain curve OA in the initial elastic range.

Motivated by these observations, we assume a special form for the response function in (13) and specify the constitutive equation for the stress by

$$s_{KL} = L_{KLMN}(e_{MN}-e^{p}_{MN}) = L_{KLMN}e^{e}_{MN} \quad , \tag{17}$$

where the coefficients L_{KLMN} are all constants. From (10) and the symmetry of s_{KL}, e_{MN} and e^{p}_{RS}, the coefficients L_{KLMN} possess the symmetries

$$L_{KLMN} = L_{LKMN} = L_{KLNM} = L_{MNKL} \quad . \tag{18}$$

The assumption (17) also provides the usual interpretation of the plastic strain as the value of the strain at a given point when the stress there is locally reduced to zero.

As the process of uniaxial deformation continues from point C in Fig. 2, the material yields in compression at point D and plastic deformation occurs continuously to point E, at which unloading again takes place. From that point, the process of cyclic loading between constant strain limits is repeated

and in most metals <u>hardening</u> (or <u>softening</u>) is observed as the stress level during each succeeding strain cycle increases (or decreases); in this connection, a hardening response is shown in Fig. 2. For most common metals, the process of cyclic loading leads to a limiting periodic response in which the stress-strain curve of each succeeding cycle is the same (see Fig. 2). This phenomenon is sometimes called <u>saturation hardening</u>, and a material is said to saturate when it reaches this limiting behavior. How rapidly saturation is reached during cyclic loading is a property of the metal.

Based upon these observations of the typical behavior of metals under cyclic loading, we assume that there is some value of the work-hardening parameter κ, say κ_s, at which $\dot{\kappa}$ is zero. Mathematically, this assumption is expressed as

$$\lim_{\kappa \to \kappa_s} \dot{\kappa} = \lim_{\kappa \to \kappa_s} h_{KL}(\mathcal{V}) \dot{e}^p_{KL} = 0 \quad . \tag{19}$$

A sufficient condition for the restriction (19) to be satisfied is that h_{KL} be homogeneous in $\kappa - \kappa_s$. In particular, we assume that h_{KL} is homogeneous of degree one in $\kappa - \kappa_s$ and is specified by*

$$h_{KL} = \left(\frac{\kappa - \kappa_s}{\kappa_o - \kappa_s}\right) \bar{h}_{KL}(s_{MN}, e^p_{MN}) \quad , \tag{20}$$

where κ_o is the initial value of κ and $\bar{h}_{KL}$ is a function of s_{MN} and e^p_{KL}. In the special case in which $\bar{h}_{KL} = s_{KL}$, combination of (6) and (20) gives

$$\dot{\kappa} = \left(\frac{\kappa - \kappa_s}{\kappa_o - \kappa_s}\right) s_{KL} \dot{e}^p_{KL} \quad . \tag{21}$$

Thus, in this special case, the rate of work-hardening $\dot{\kappa}$ is proportional to the

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*Since the experimental observations that motivate this assumption do not depend on the deformation being small, the special form (20) of the hardening response function may also be viewed in the context of finite deformation.

rate of nonrecoverable work $s_{KL}\dot{e}^p_{KL}$ in the material. Since a notion of this kind is appealing on physical grounds, it is desirable to have the linearized version of (6) include (21) as a special case. Hence, we assume that h_{KL} is order ϵ and sufficiently smooth so that it may be expressed as a linear function of s_{KL} and e^p_{KL}. This leads us to write

$$\bar{h}_{KL} = M_{KLMN}s_{MN} + N_{KLMN}e^p_{MN} \quad , \tag{22}$$

where the constants M_{KLMN} and N_{KLMN} possess obvious symmetries.

The foregoing assumptions for the stress response (17) and the rate of work-hardening (20) with $\bar{h}_{KL}$ given by (22) are motivated by observations of the behavior of metals in one dimension. From the same information, however, it is not immediately clear what may be an appropriate representation for the loading function in the infinitesimal theory. In the present paper, we assume that f is sufficiently smooth so that it may be represented as a polynomial in s_{KL}, e^p_{MN} and κ. In addition, we note that for (4) to represent a closed surface in stress space, f must be at least quadratic in the stress. It follows that f must be at least second order in ϵ. And whatever the order of f, all terms in its polynomial representation must be the same order in ϵ. Since the ratio $(\kappa-\kappa_s)/(\kappa_o-\kappa_s)$ is order 1, it follows from (6), (20) and (22) that $\dot{\kappa}$ and hence also κ are order ϵ^2. Thus, the loading function must satisfy the relation

$$f(\epsilon s_{KL}, \epsilon e^p_{MN}, \epsilon^2\kappa) = \epsilon^n f(s_{KL}, e^p_{MN}, \kappa) \tag{23}$$

when

$$f = O(\epsilon^n) \quad , \quad n \geqq 2 \quad . \tag{24}$$

Because for each value of the integer n the loading function has a different general representation, there may be any number of possible forms of the elastic-plastic constitutive equations, each corresponding to a different n, which are

compatible with the assumption of small strain. Examples of two commonly used yield functions that correspond to different orders of f are the Tresca and von Mises yield conditions. Expressed as a polynomial in s_{KL}, the Tresca yield condition corresponds to $n = 6$ while the von Mises condition corresponds to $n = 2$.

Because we wish to describe the behavior of metals and since the von Mises yield condition has been experimentally shown to be a suitable representation for the initial yield surface in isotropic metals,* we assume f to be second order in ϵ. In this case the general representation for f is (see Ref. [1], Eq. (10.11))

$$f = B_{KLMN}s_{KL}s_{MN} + E_{KLMN}e^p_{KL}e^p_{MN} + F_{KLMN}s_{KL}e^p_{MN} - \kappa \quad , \tag{25}$$

where the constant coefficients B_{KLMN}, E_{KLMN} and F_{KLMN} have obvious symmetries and without loss in generality we have taken the coefficient of κ to be 1. Naghdi and Trapp [15] have shown that for a stress response in the form (17), convexity of the loading surface in stress space follows from a work inequality. This requirement may place certain restrictions on the coefficients in (25).

When the stress is specified in the form (17), the relation (14) for the rate of plastic strain reduces to (see Ref. [15], Eq. (24))

$$\dot{e}^p_{KL} = \gamma \frac{\partial f}{\partial s_{KL}} \quad , \quad \gamma > 0 \quad . \tag{26}$$

Since $\hat{f} > 0$ during loading and $\lambda > 0$ it is possible to identify

$$\gamma = \lambda \hat{f} \tag{27}$$

and a comparison of $(5)_4$ with (27) yields

$$\beta_{KL} = \frac{\partial f}{\partial s_{KL}} \quad . \tag{28}$$

*For example, see Taylor and Quinney [17], Naghdi et al. [18], or Bertsch and Findley [19].

The constitutive equations developed in this section are appropriate for small deformations of a general anisotropic metal. The case of an initially isotropic metal is discussed in the next section.

4. Isotropic Metals.

We now consider a special case of the infinitesimal theory appropriate for metals which are initially isotropic. Before doing this, it is convenient to introduce some additional notation. Let s_{KL} and e^p_{KL} be decomposed in the form

$$
\begin{aligned}
s_{KL} &= \bar{s}\delta_{KL} + \tau_{KL} \quad , \quad \bar{s} = \frac{1}{3} s_{MM} \quad , \\
e^p_{KL} &= \bar{e}^p \delta_{KL} + \gamma^p_{KL} \quad , \quad \bar{e}^p = \frac{1}{3} e^p_{MM} \quad ,
\end{aligned}
\tag{29}
$$

where τ_{KL} and γ^p_{KL} are the deviatoric parts of the stress and the plastic strain, respectively, and $\bar{s}$ denotes the <u>mean normal stress</u>.

For an initially isotropic material, L_{KLMN} reduces to an isotropic fourth order tensor and the stress response (17), expressed in inverted form, is given by

$$
e^e_{KL} = \frac{1+\nu}{E} s_{KL} - \frac{\nu}{E} s_{MM}\delta_{KL} \quad , \tag{30}
$$

where E and ν are Young's modulus and Poisson's ratio, respectively. Similarly, the coefficients in the expression (22) for $\bar{h}_{KL}$ reduce to fourth order isotropic tensors and hence this response function can be expressed in the form

$$
\bar{h}_{KL} = \alpha_1 s_{MM}\delta_{KL} + \alpha_2 e^p_{MM}\delta_{KL} + \alpha_3 s_{KL} + \alpha_4 e^p_{KL} \quad , \tag{31}
$$

where $\alpha_1, \alpha_2, \alpha_3$ and α_4 are constants. In addition, the loading function is restricted to be of the form

$$
\begin{aligned}
f = {} & \beta_1 (s_{MM})^2 + \beta_2 (s_{KL}s_{KL}) + \beta_3 (e^p_{MM})^2 + \beta_4 (e^p_{KL}e^p_{KL}) \\
& + \beta_5 (s_{MM}e^p_{NN}) + \beta_6 (s_{KL}e^p_{KL}) - \kappa
\end{aligned}
\tag{32}
$$

where the coefficients $\beta_1, \ldots, \beta_6$ are all constants.

In the well-known experiments of Bridgeman [20], a large number of metals were subjected to tensile tests under varying amounts of hydrostatic pressure.

He observed that the initial yield point of each metal was not altered by the presence of the added pressure and that the character of the stress-strain curve during subsequent plastic deformation remained unchanged. In the present paper, we use the first of these observations to place a restriction on the form of the loading function (32).

Prior to any plastic deformation, $e^p_{KL} = 0$ and (32) reduces to

$$f(s_{KL}, 0, \kappa_o) = (3\beta_1 + \beta_2)\bar{s}^2 + \beta_2 \tau_{KL}\tau_{KL} - \kappa_o \quad , \tag{33}$$

where we have used the decomposition $(29)_1$. Motivated by the results of Bridgeman's experiments, we assume that before any plastic deformation has occurred, the yield function is independent of the mean normal stress. Note that this restriction is confined to the form of the <u>initial</u> yield function and no assumption is made regarding subsequent plastic deformation. From this assumption and (33) we conclude that

$$\frac{\partial}{\partial \bar{s}} [f(s_{KL}, 0, \kappa_o)] = 0 \quad \text{or} \quad 3\beta_1 + \beta_2 = 0 \tag{34}$$

and hence (32) may be written in the form

$$\begin{aligned} f = {} & \tau_{KL}\tau_{KL} + \beta_3 (e^p_{MM})^2 + \beta_4 (e^p_{KL} e^p_{KL}) \\ & + \beta_5 (s_{MM} e^p_{NN}) + \beta_6 (s_{KL} e^p_{KL}) - \kappa \quad , \end{aligned} \tag{35}$$

where by properly adjusting κ_o, we have taken $\beta_2 = 1$. When the loading function is specified in the form (35), the corresponding expression for the plastic strain rate can be computed from (26) and is given by

$$\dot{e}^p_{KL} = \gamma [2\tau_{KL} + \beta_5 e^p_{MM} \delta_{KL} + \beta_6 e^p_{KL}] \quad . \tag{36}$$

Contracting this expression on the indices K,L and using $(29)_2$, we obtain

$$\dot{\bar{e}}^p = \gamma (3\beta_5 + \beta_6) \bar{e}^p \quad . \tag{37}$$

If we assume that during loading γ is a continuous function of time, the unique solution of (37) which is compatible with a zero initial value for $\bar{e}^p$ is the trivial solution

$$\bar{e}^p = e^p_{MM} = 0 \quad . \tag{38}$$

Hence the plastic volume change is identically zero.‡ This result follows directly from $(34)_1$ and (26) and need not be assumed independently.

The expression (35) for f simplifies as a result of (38), and the rate of work-hardening may now be expressed as

$$\dot{\kappa} = \left(\frac{\kappa-\kappa_s}{\kappa_o-\kappa_s}\right)\bar{h}_{KL}\dot{\gamma}^p_{KL} \quad . \tag{39}$$

Hence, we need only retain the deviatoric part of h_{KL}. In summary then, the loading function and hardening response are given by the reduced forms

$$\begin{aligned} f(s_{KL},e^p_{KL},\kappa) &= \bar{f}(\tau_{KL},\gamma^p_{KL},\kappa) \\ &= \tau_{KL}\tau_{KL} - \alpha\tau_{KL}\gamma^p_{KL} + \sigma\gamma^p_{KL}\gamma^p_{KL} - \kappa \quad , \\ \dot{\kappa} &= \frac{\kappa-\kappa_s}{\kappa_o-\kappa_s}\,[\beta\tau_{KL} + \eta\gamma^p_{KL}]\dot{\gamma}^p_{KL} \quad , \end{aligned} \tag{40}$$

where we have set

$$-\beta_6 = \alpha \quad , \quad \alpha_3 = \beta \quad , \quad \beta_4 = \sigma \quad , \quad \alpha_4 = \eta \quad . \tag{41}$$

Apart from the term $\sigma(\gamma^p_{KL}\gamma^p_{KL})$, which is independent of the stress and hence effectively plays the same role as κ in determining the form of the yield surface in stress space, the form of the loading function $(40)_1$ is the same as one proposed previously by Edelman and Drucker [22]. As explained in their paper, a yield function in the form $(40)_1$ exhibits a Bauschinger effect and reduces to the von Mises yield condition in the absence of plastic strain.

We close this section by identifying two well-known hardening rules.

‡We emphasize that the restriction (38) is not an essential ingredient of plasticity theory even for small deformation. It may be imposed by special assumption or, as in the present development, it follows from a particular derivation. In this connection, see also [21].

namely the isotropic and the kinematic hardening rules, as special cases of the above results.* First, when

$$\eta = \alpha = \sigma = 0 \text{ and in the limit as } \kappa_s \to \infty \quad , \tag{42}$$

the expressions $(40)_{1,2}$ reduce to

$$\bar{f} = \tau_{KL}\tau_{KL} - \kappa \quad , \quad \dot{\kappa} = \beta\tau_{KL}\dot{\gamma}^{p}_{KL} \quad . \tag{43}$$

This is a form of <u>isotropic</u> <u>hardening</u>. Next, if we set

$$\eta = \beta = 0 \quad , \quad 4\sigma = \alpha^2 \quad , \tag{44}$$

the expressions $(40)_{1,2}$ become

$$\bar{f} = (\tau_{KL} - \frac{\alpha}{2}\gamma^{p}_{KL})(\tau_{KL} - \frac{\alpha}{2}\gamma^{p}_{KL}) - \kappa \quad , \quad \dot{\kappa} = 0 \text{ (or } \kappa = \kappa_0 = \text{const.)} \quad . \tag{45}$$

This is the usual form of <u>kinematic</u> <u>hardening</u>.

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*An account of these hardening rules may be found in [6].

5. Restrictions on the coefficients in the yield function and hardening response.

So far we have used the elastic-plastic constitutive equations formulated relative to a loading surface in stress space because this form made it more convenient to appeal to certain physical observations of metal behavior. However, since these constitutive equations have certain limitations*, we consider further developments in the context of strain space. Before recording the corresponding forms of $(40)_{1,2}$ relative to a loading surface in strain space, however, we need to recall from [3] certain results in terms of the list of variables $(9)_2$.

For fixed values of e^p_{KL} and κ, the loading surface in strain space is specified by

$$g(e_{KL}, e^p_{MN}, \kappa) = 0 \quad . \tag{46}$$

Relative to g, the constitutive equations for $\dot{e}^p_{KL}$ and $\dot{\kappa}$ are expressed as

$$\dot{e}^p_{KL} = \begin{cases} 0 & \text{when } g < 0 \quad , \\ 0 & \text{when } g = 0 \text{ and } \hat{g} \leq 0 \quad , \\ \bar{\lambda}\rho_{KL}\hat{g} & \text{when } g = 0 \text{ and } \hat{g} > 0 \quad , \end{cases} \tag{47}$$

and

$$\dot{\kappa} = m_{KL}\dot{e}^p_{KL} \quad , \tag{48}$$

where

$$\hat{g} = \frac{\partial g}{\partial e_{KL}} \dot{e}_{KL} \quad . \tag{49}$$

The positive scalar function $\bar{\lambda}$ and the symmetric second order tensor functions ρ_{KL} and m_{KL} depend on

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*We do not elaborate here on these limitations which are discussed by Naghdi and Trapp [3].

$$u = (e_{KL}, e^p_{MN}, \kappa) \quad . \tag{50}$$

Following [3], we assume that g is obtained from f through the expression

$$g(e_{KL}, e^p_{MN}, \kappa) = f(\hat{s}_{KL}(e_{PQ}, e^p_{RS}, \kappa), e^p_{MN}, \kappa) \tag{51}$$

and that m_{KL} is obtained from h_{KL} through the same substitution in the arguments of h_{KL}.

To ensure that succeeding values of strain remain on the yield surface (51) during loading, we must have $\dot{g} = 0$ whenever $\hat{g} > 0$. This condition, together with (46) and (49), leads to

$$\bar{\lambda} \rho_{KL} \left(\frac{\partial g}{\partial e^p_{KL}} + m_{KL} \frac{\partial g}{\partial \kappa} \right) = -1 \quad . \tag{52}$$

A condition analogous to (52) also arises in the stress space formulation. From (52) and the sign of $\bar{\lambda}$ follows the restriction

$$\rho_{KL} \left(\frac{\partial g}{\partial e^p_{KL}} + m_{KL} \frac{\partial g}{\partial \kappa} \right) < 0 \quad , \tag{53}$$

which holds at all times during <u>loading</u>. With the help of (51), the corresponding form of (12) relative to strain space is

$$\left(\frac{\partial \hat{s}_{KL}}{\partial e^p_{MN}} + \frac{\partial \hat{s}_{KL}}{\partial \kappa} m_{MN} \right) \dot{e}^p_{MN} = - \gamma \frac{\partial g}{\partial e_{KL}} \quad . \tag{54}$$

To recast $(40)_{1,2}$ in the corresponding forms relative to strain space, we first take the deviatoric part of the assumed stress response (30):

$$\tau_{KL} = 2\mu \gamma^e_{KL} \quad , \tag{55}$$

where γ^e_{KL} is the deviatoric part of e^e_{KL} and μ is the shear modulus of elasticity. Combining (51), (55) and $(40)_{1,2}$, we then have

$$g(e_{KL},e^p_{MN},\kappa) = \bar{g}(\gamma_{KL},\gamma^p_{KL},\kappa)$$
$$= 4\mu^2\gamma^e_{KL}\gamma^e_{KL} - 2\mu\alpha\gamma^e_{KL}\gamma^p_{KL} + \sigma\gamma^p_{KL}\gamma^p_{KL} - \kappa \quad , \tag{56}$$
$$m_{KL} = \frac{\kappa-\kappa_s}{\kappa_o-\kappa_s}(2\mu\beta\gamma^e_{KL} + \eta\gamma^p_{KL}) \quad .$$

Substitution of (55) into (54) yields

$$2\mu\dot{\gamma}^p_{KL} = \gamma\frac{\partial g}{\partial e_{KL}} = \gamma\frac{\partial\bar{g}}{\partial\gamma_{KL}} \quad , \tag{57}$$

where the first of (56) has been used to obtain $(57)_2$. Now let

$$\gamma = 2\mu\overline{\lambda \hat{g}} \quad , \tag{58}$$

so that from (47) and (58) we may identify

$$\rho_{KL} = \frac{\partial\bar{g}}{\partial\gamma_{KL}} \quad . \tag{59}$$

The form of the elastic-plastic constitutive equations derived in section 4 for metals has now been completely determined relative to strain space. It remains to examine the consequences of the restriction (53). Substitution of (56) and (59) into (53) yields the inequality

$$2\mu\{[4\mu\gamma^e_{KL}-\alpha\gamma^p_{KL}][(\bar{\beta}+\alpha+4\mu)2\mu\gamma^e_{KL} + (\bar{\eta}-2\mu\alpha-2\sigma)\gamma^p_{KL}]\} > 0 \quad , \tag{60}$$

where for convenience we have put

$$\bar{\beta} = (\frac{\kappa-\kappa_s}{\kappa_o-\kappa_s})\beta \quad , \quad \bar{\eta} = (\frac{\kappa-\kappa_s}{\kappa_o-\kappa_s})\eta \quad . \tag{61}$$

The restriction (60) holds at all times during <u>loading</u>. Since it does not involve rate quantities, (60) must then hold for all values of $\gamma_{KL},\gamma^p_{KL}$ and κ that satisfy $g=0$. With the help of $(56)_1$, $g=0$ can be expressed as

$$M_{KL}M_{KL} = \kappa - (\sigma - \frac{\alpha^2}{4})\gamma^p_{KL}\gamma^p_{KL} \quad , \tag{62}$$

where M_{KL} is the deviatoric second order tensor defined by

$$M_{KL} = 2\mu\gamma^{e}_{KL} - \tfrac{1}{2}\,\alpha\gamma^{p}_{KL} \quad . \tag{63}$$

For $\gamma_{KL}, \gamma^{p}_{KL}$ and κ to satisfy $g = 0$, it is sufficient that M_{KL} satisfy (62). Thus M_{KL} is fixed in magnitude, but otherwise unrestricted. Combination of (63) and (60) gives

$$4\mu M_{KL}\{(\bar{\beta}+\alpha+4\mu)M_{KL} + [\tfrac{1}{2}\alpha(\bar{\beta}+\alpha) + \bar{\eta} - 2\sigma]\gamma^{p}_{KL}\} > 0 \quad . \tag{64}$$

which must hold for all γ^{p}_{KL} and κ, and all M_{KL} that satisfy (62). It follows from the inequality (64) that†

$$\bar{\beta} + \alpha + 4\mu > 0 \tag{65}$$

and

$$(\bar{\beta}+\alpha+4\mu)^2 M_{KL}M_{KL} > [\tfrac{1}{2}\alpha(\bar{\beta}+\alpha) + \bar{\eta} - 2\sigma]^2 \gamma^{p}_{KL}\gamma^{p}_{KL} \quad . \tag{66}$$

Since (65) holds for all values of κ and $0 < (\kappa-\kappa_s)/(\kappa_o-\kappa_s) \leq 1$, the coefficients α and β must satisfy

$$\alpha + 4\mu > 0 \quad , \quad \alpha + \beta + 4\mu > 0 \quad . \tag{67}$$

Substitution of (62) into (66) gives

$$(\bar{\beta}+\alpha+4\mu)^2\kappa > \{(\bar{\beta}+\alpha+4\mu)^2(\sigma - \frac{\alpha^2}{4}) + [\tfrac{1}{2}\alpha(\bar{\beta}+\alpha) + \bar{\eta} - 2\sigma]^2\}\gamma^{p}_{KL}\gamma^{p}_{KL} \quad . \tag{68}$$

For a given material, the inequality (68) must hold for all admissible values of the scalars κ and $\gamma^{p}_{KL}\gamma^{p}_{KL}$: κ is bounded by κ_o and κ_s, and $\gamma^{p}_{KL}\gamma^{p}_{KL}$ cannot become unbounded before the material fractures. Thus, the range of admissible values for κ and $\gamma^{p}_{KL}\gamma^{p}_{KL}$ depends on the particular material and a general development of the type considered here can proceed no further than (68).

† For details of these calculations, see Appendix A at the end of the paper.

A <u>sufficient</u> condition that (68) be satisfied for <u>all</u> κ and γ^p_{KL} is that the right-hand side of (66) vanish identically, i.e.,

$$(\tfrac{1}{2}\alpha\bar{\beta}+\bar{\eta}) + (\tfrac{1}{2}\alpha^2-2\sigma) = 0 \quad . \tag{69}$$

Since this equation must hold for all admissible values of κ, it follows that

$$4\sigma = \alpha^2 \quad \text{and} \quad \alpha\beta + 2\eta = 0 \quad . \tag{70}$$

Substitution of these restrictions into $(56)_{1,2}$ yields the reduced expressions

$$g = M_{KL}M_{KL} - \kappa \quad , \quad \dot{\kappa} = \beta\left(\frac{\kappa-\kappa_s}{\kappa_o-\kappa_s}\right)M_{KL}\dot{\gamma}^p_{KL} \quad . \tag{71}$$

These restricted constitutive equations satisfy the inequality (53) identically provided α and β satisfy $(67)_{1,2}$. We emphasize, however, that $(70)_{1,2}$ are merely sufficient conditions that (68) be satisfied; the only <u>necessary</u> restrictions on the constitutive coefficients for a general material are $(67)_{1,2}$.

A procedure completely analogous to the one employed in this section can be used to derive restrictions on the coefficients α,β,σ and η in the stress space formulation of the basic equations. Similar results follow except that α and β satisfy the more stringent conditions

$$\alpha > 0 \quad , \quad \alpha + \beta > 0 \tag{72}$$

and the full set of coefficients satisfy

$$(\bar{\beta}+\alpha)^2\kappa > \{(\bar{\beta}+\alpha)^2(\sigma - \frac{\alpha^2}{4}) + [\tfrac{1}{2}\alpha(\bar{\beta}+\alpha) + \bar{\eta} - 2\sigma]\}\gamma^p_{KL}\gamma^p_{KL} \tag{73}$$

for all admissible values of* κ and $\gamma^p_{KL}\gamma^p_{KL}$. The restrictions (72) and (73) differ from $(67)_{1,2}$ and (68) because the conditions for <u>loading</u> in each formulation are not equivalent. To see this, combine (51), (55) and $(40)_1$ to obtain

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*Note that the conditions $(70)_{1,2}$ are also sufficient that (73) be satisfied for all κ and γ^p_{KL}, just as in the strain space formulation.

$$\hat{f} = \frac{\partial \bar{f}}{\partial \tau_{KL}} \dot{\tau}_{KL} = \frac{\partial \bar{g}}{\partial \gamma_{KL}} \dot{\gamma}^{e}_{KL} = \hat{g} - \frac{\partial \bar{g}}{\partial \gamma_{KL}} \dot{\gamma}^{p}_{KL} \quad . \tag{74}$$

Since $\dot{\gamma}^{p}_{KL} \neq 0$ during loading, $\hat{g} > 0$ does not necessarily mean that $\hat{f} > 0$. Further discussion of this point can be found in [3].

6. Determination of material coefficients: uniaxial cyclic loading.

In general, there are six material constants

$$\alpha, \beta, \sigma, \eta, \kappa_o, \kappa_s \quad , \tag{75}$$

which must be determined by experiment. For the restricted constitutive equations $(71)_{1,2}$, the number of independent constants is reduced to four since in that case σ and η are specified by $(70)_{1,2}$. A procedure for determining these constants from experiments in uniaxial cyclic loading is outlined in this section.

Consider first a homogeneous extensional deformation in which the only nonvanishing component of the stress tensor is $s_{11} = s(t)$ say. Then, from (29), (30), (38) and the symmetry properties for isotropic materials, we have

$$\tau_{KL} = \begin{Vmatrix} \frac{2}{3}s & 0 & 0 \\ 0 & -\frac{1}{3}s & 0 \\ 0 & 0 & -\frac{1}{3}s \end{Vmatrix} , \quad \gamma^p_{KL} = e^p_{KL} = \begin{Vmatrix} e_p & 0 & 0 \\ 0 & -\frac{1}{2}e_p & 0 \\ 0 & 0 & -\frac{1}{2}e_p \end{Vmatrix} , \tag{76}$$

$$s = E(e_{11} - e^p_{11}) = E(e - e_p) \quad , \tag{77}$$

where the notations e and e_p are introduced for convenience. Corresponding to the above homogeneous deformation, the loading functions f and g assume the simplified forms

$$f = \frac{2}{3}s^2 - \alpha s e_p + \frac{3}{2}\sigma e_p^2 - \kappa \quad , \tag{78}$$

$$g = \frac{2}{3}E^2(e-e_p)^2 - \alpha E(e-e_p)e_p + \frac{3}{2}\sigma e_p^2 - \kappa \quad . \tag{79}$$

Also, during loading, the expression for $\dot{e}_p$ is

$$\dot{e}_p = \frac{\frac{2}{3}E(\frac{4}{3}s - \alpha e_p)\dot{e}}{\frac{2}{3}(\bar{\beta} + \alpha + \frac{4}{3}E)s + (\bar{\eta} - \frac{2}{3}E\alpha - 2\sigma)e_p} \quad , \tag{80}$$

where in obtaining (80) we have used (52), (57), (58) and (79). Since the deformation is homogeneous, we can obtain the slope of the stress-strain curve during loading by combining (77) and (80). This calculation gives

$$\frac{ds}{de} = \left[\frac{1}{E} + \frac{\frac{4}{3}\, s - \alpha e_p}{(\bar{\beta}+\alpha)s + 3(\bar{\eta}/2 - \sigma)e_p}\right]^{-1} \quad . \tag{81}$$

The above expression at the onset of initial yield reduces to

$$\left.\frac{ds}{de}\right|_{e_p=0} = \left[\frac{1}{E} + \frac{4/3}{\beta+\alpha}\right]^{-1} \tag{82}$$

and in the limit of saturation hardening the slope becomes

$$\lim_{\kappa \to \kappa_s} \frac{ds}{de} = \left[\frac{1}{E} + \frac{\frac{4}{3}\, s - \alpha e_p}{\alpha s - 3\sigma e_p}\right]^{-1} \quad . \tag{83}$$

Equations (80) and (81) hold for all metallic materials without restriction, except that the deformation is assumed to be infinitesimal and homogeneous. If we impose the conditions $(70)_{1,2}$, then these expressions reduce to

$$\dot{e}_p = \frac{\frac{4}{3}\, E\dot{e}}{(\bar{\beta} + \alpha + \frac{4}{3}\, E)} \quad , \quad \frac{ds}{de} = \left[\frac{1}{E} + \frac{\frac{4}{3}}{\bar{\beta} + \alpha}\right]^{-1} \quad . \tag{84}$$

Hence, the slope of the stress-strain curve derived from the restricted constitutive equations $(71)_{1,2}$ is independent of both s and e_p, but is not constant since $\bar{\beta}$ depends on κ. During loading the slope $(84)_2$ decreases from a value of

$$\left[\frac{1}{E} + \frac{\frac{4}{3}}{\beta + \alpha}\right]^{-1} \tag{85}$$

at the onset of initial yield, to the value

$$\left[\frac{1}{E} + \frac{4}{3\alpha}\right]^{-1} \tag{86}$$

at saturation.

Consider now a uniaxial elastic-plastic state specified by

$$s = \overset{*}{s}_{(1)} \quad , \quad e_p = \overset{*}{e}_p \quad , \quad \kappa = \overset{*}{\kappa} \tag{87}$$

and suppose that the material is unloaded from this state until it yields again in the reverse direction at $s = \overset{*}{s}_{(2)}$. Then, since both $\overset{*}{s}_{(1)}$ and $\overset{*}{s}_{(2)}$ are roots of (78) for the same values of e_p and κ, we have

$$\overset{*}{s}_{(1)} + \overset{*}{s}_{(2)} = \frac{3}{2}\alpha \overset{*}{e}_p \tag{88}$$

so that

$$\alpha = \frac{2}{3} \frac{\overset{*}{s}_{(1)} + \overset{*}{s}_{(2)}}{\overset{*}{e}_p} \quad . \tag{89}$$

Experimental data at a number of stress reversals can be used in (89) to obtain an estimate of α, and this value can then be used in (82) to determine β from the slope of the stress-strain curve just after initial yield.[†] Next, σ is determined from (83) together with measured slopes of the stress-strain curve at various loading points following saturation. A value for κ_o follows from the initial yield stress, and κ_s is computed from (78) together with experimental values of stress and plastic strain at saturation. Since η does not enter the yield function and it drops out of both limiting slopes (82) and (83), we must determine its value by using the general expression (81). Hence we need to measure the slope of the stress-strain curve at several loading points between initial yield and saturation. The value of κ at such a loading point is computed from (78). This value is then used in (81) to estimate an appropriate value for η.

The above procedure is suggested as one means of determining the six

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[†]When using this procedure, comments in the Introduction (Section 1) regarding the idealized transition from elastic to plastic deformation should be kept in mind. The gradual transition which is observed experimentally should be idealized by a discontinuous slope in the stress-strain curve (see Fig. 1) before the data is used to determine the material coefficients.

material coefficients from cyclic loading data; by no means is the procedure unique. If additional experimental information is available, other ways of determining the coefficients may be more convenient.

7. Comparison with experiments.

As an example of a metal that may be adequately represented by the restricted constitutive equations (71), consider the cyclic stress-strain behavior of 304 stainless steel reported by Pugh et al. [13]. In this study, specimens were cycled in tension and compression at $649^{\circ}C$ ($1200^{\circ}F$) between strain limits of $\pm$ 0.01 until saturation was reached. The reported results [13] are reproduced here in Fig. 3(a). Using the procedure outlined in Section 6, together with the data of Fig. 3(a), one is able to choose the four material constants in the restricted constitutive equations $(71)_{1,2}$ as*

$$\frac{\alpha}{E} = 4.06 \times 10^{-3} \quad , \quad \frac{\beta}{E} = 39 \times 10^{-3} \quad ,$$
$$\frac{\kappa_o}{E^2} = 0.54 \times 10^{-6} \quad , \quad \frac{\kappa_s}{E^2} = 2.46 \times 10^{-6} \quad , \tag{90}$$

where the elastic modulus E is§

$$E = 123 \text{ GPa (or } 17.8 \times 10^{6} \text{ psi)} \quad . \tag{91}$$

Theoretical stress-strain curves were determined by explicit numerical integration of (80) using the values (90) and (91) for the material coefficients. Results of this computation are shown in Fig. 3(b); and, on the same graph, the corresponding experimental curves from Fig. 3(a) are shown for the first two strain cycles. The comparison for additional strain cycles is not exhibited in Fig. 3(b) simply because it would crowd the figure.

Consistent with the experimental observations in [13], the theory predicts that saturation occurs after about four strain cycles. This should be contrasted with Eisenberg's [12] development which shows at least eight cycles before effective saturation. This occurs in spite of his more detailed treatment of transition from elastic to plastic deformation.

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*Computations leading to these values are given in Appendix B at the end of the paper.

§The abbreviation GPa stands for Giga Pascal = 10^9 Pascal = 10^9 Newton/m^2.

In order to further examine the applicability of the theoretical developments of the present paper, we conducted cyclic tension-compression tests on cylindrical specimens of 2024-T351 aluminum alloy (with solid cross-sections) machined from plate stock.* The specimen had a gauge length of 2.54 cm. (1 in.) and a diameter of 1 cm. (0.375 in.). Each specimen was locked in threaded fixtures and loaded in a model TTC SP Instron testing machine at a nominal crosshead speed of (0.005 in/min). Strain was measured with an Instron clip-on extensometer.

Two different loading programs were used, each nonsymmetric with respect to zero strain. The results of the two experiments are shown as dashed lines in figures 4 and 5. At least three specimens were used in each case to ensure repeatability of the data. Since relatively high stress levels were reached during the first loading cycle and because the material hardened rather strongly, the tests were stopped after two loading cycles to avoid buckling the specimen.

In order to adequately represent the observed behavior of the 2024 aluminum alloy, it was necessary to use the unrestricted constitutive equations $(56)_{1,2}$. The condition $(70)_2$ was retained, however, for simplicity.† Values for the five independent coefficients were selected by fitting the data shown by the dashed lines in Fig. 4. Although the available experimental data in either loading program is sufficient to determine α, β and κ_0 according to the procedure in section 6, the remaining constants σ and κ_s are difficult to determine directly since the experiments did not proceed to saturation. Consequently, σ and κ_s were adjusted until the predicted stress-strain curves under the loading program in Fig. 4 showed reasonable agreement with the experimental data. The results are:

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*The tests were carried out in the laboratories of the Dept. of Mechanical Engineering of the University of California, Berkeley, using a standard Instron testing machine.

†Recall that the conditions $(70)_{1,2}$ were only <u>sufficient</u> for satisfaction of the inequality (53).

$$\frac{\alpha}{E} = 0.07 \quad , \quad \frac{\beta}{E} = 0.023 \quad , \quad \frac{\sigma}{E^2} = 8 \times 10^{-3} \quad ,$$
$$\frac{\kappa_o}{E^2} = 18 \times 10^{-6} \quad , \quad \frac{\kappa_s}{E^2} = 30 \times 10^{-6} \quad , \tag{92}$$

and

$$E = 69 \text{ GPa (or } 10^7 \text{ psi)} \quad . \tag{93}$$

The values (92) and (93) were then used to compute the stress-strain curves corresponding to the second loading program shown in Fig. 5. The predicted stress-strain curves are shown in both Figs. 4 and 5 for comparison. It is evident that the nature of the agreement is similar in each case. Also, it is interesting to note that the reasonably good agreement with the behavior of 2024 aluminum alloy was achieved at moderate values for strain, even though the theoretical developments were derived for small deformation.

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Appendix A

We provide here details of the calculations leading to the inequalities (65) and (66). For this purpose, it is convenient to regard M_{KL} and γ^p_{KL} as vectors in the five-dimensional subspace defined by the hypersurface $e_{MM} = 0$ in six-dimensional strain space and write (62) and (64) in the forms

$$\underset{\sim}{M} \cdot \underset{\sim}{M} = c^2 \quad , \quad \underset{\sim}{M} \cdot (a\underset{\sim}{M} + b\underset{\sim}{\gamma}^p) > 0 \quad , \tag{A1}$$

where

$$\begin{aligned} a &= \bar{\beta} + \alpha + 4\mu \quad , \quad b = \tfrac{1}{2}\,\alpha(\bar{\beta}+\alpha) + \bar{\eta} - 2\sigma \quad , \\ c^2 &= \kappa - (\sigma - \frac{\alpha^2}{4})\gamma^p_{KL}\gamma^p_{KL} \quad . \end{aligned} \tag{A2}$$

First, choose $\underset{\sim}{M}$ such that $\underset{\sim}{M} \cdot \underset{\sim}{\gamma}^p = 0$. Then, $(A1)_2$ implies that $a > 0$ and the inequality (65) follows immediately. Next, write the inequality $(A1)_2$ as

$$a(\underset{\sim}{M} \cdot \underset{\sim}{M}) + b(\underset{\sim}{M} \cdot \underset{\sim}{\gamma}^p) > 0 \quad . \tag{A3}$$

The first term in (A3) is nonnegative, in view of (65). Then, by the Schwartz inequality, namely $|\underset{\sim}{M} \cdot \underset{\sim}{\gamma}^p| \leqq |\underset{\sim}{M}||\underset{\sim}{\gamma}^p|$, and the restriction $(A1)_1$, the second term in (A3) satisfies

$$b(\underset{\sim}{M} \cdot \underset{\sim}{\gamma}^p) \geqq -|b|(\underset{\sim}{M} \cdot \underset{\sim}{M})^{\frac{1}{2}}(\underset{\sim}{\gamma}^p \cdot \underset{\sim}{\gamma}^p)^{\frac{1}{2}} = -|bc|(\underset{\sim}{\gamma}^p \cdot \underset{\sim}{\gamma}^p)^{\frac{1}{2}} \quad . \tag{A4}$$

In order that (A3) hold for all values of $\underset{\sim}{\gamma}^p$ and for all $\underset{\sim}{M}$ satisfying $(A1)_1$, it is both necessary and sufficient that $a(\underset{\sim}{M} \cdot \underset{\sim}{M}) - |bc|(\underset{\sim}{\gamma}^p \cdot \underset{\sim}{\gamma}^p)^{\frac{1}{2}} > 0$ or

$$a^2(\underset{\sim}{M} \cdot \underset{\sim}{M})^2 > b^2c^2(\underset{\sim}{\gamma}^p \cdot \underset{\sim}{\gamma}^p) \quad . \tag{A5}$$

Next, with the use of $(A1)_1$, we obtain

$$a^2(\underset{\sim}{M} \cdot \underset{\sim}{M}) > b^2(\underset{\sim}{\gamma}^p \cdot \underset{\sim}{\gamma}^p) \qquad \text{(A6)}$$

from which and $(A2)_{1,2}$ follows the result (66).

Appendix B

The material coefficients for the 304 stainless steel in (90) were obtained by the procedure outlined in section 6. This appendix includes details of the calculations.

First, when (89) is applied to the several stress reversals of Fig. 3(a) which go from tension to compression, α is consistently calculated as

$$\alpha \cong -0.5 \text{ GPa} \quad . \tag{B1}$$

On the other hand, the stress reversals which go from compression to tension consistently give

$$\alpha \cong 1.5 \text{ GPa} \quad . \tag{B2}$$

The difference between these two values, while seemingly large relative to α, represents only small fluctuations in $s^*_{(1)}$ and $s^*_{(2)}$ [see Equations (89)] and may be due to a slight material anisotropy. We choose the average value

$$\alpha = 0.5 \text{ GPa} \quad . \tag{B3}$$

A value for β is obtained from the slope of the stress-strain curve just after initial yield. From (82) and Fig. 3(a)

$$\left.\frac{ds}{de}\right|_{e_p=0} = 3.85 \text{ GPa} = \left[\frac{1}{E} + \frac{4/3}{\beta+\alpha}\right]^{-1} \tag{B4}$$

and hence (B3), (B4) and (91) give

$$\beta = 0.477 \text{ GPa} \quad . \tag{B5}$$

Next, κ_o is computed from (78), evaluated at $e_p = 0$. Based on an initial yield stress of 110 MPa, we obtain

$$\kappa_o = 0.0081 \ (\mathrm{GPa})^2 \quad . \tag{B6}$$

A value for κ_s follows from experimental data at saturation. For example, after saturation and just before unloading from tension, we have

$$s = 248 \ \mathrm{MPa} \quad , \quad e_p = 0.0081 \quad . \tag{B7}$$

Substitution of these variables into (78) yields

$$\kappa_s = 0.037 \ (\mathrm{GPa})^2 \quad . \tag{B8}$$

Appropriate normalization of (B3), (B5), (B6) and (B8) with respect to the elastic modulus in (91) yields the numerical values given in (90).

Captions for Figures

Fig. 1 Mechanical response of a typical ductile metal under uniaxial loading with theoretical idealizations indicated by dashed lines (the symbols s and e stand for one-dimensional components of stress and strain): Fig. 1(a) exhibits loading in simple tension followed by unloading and reloading in tension, while Fig. 1(b) exhibits loading in simple tension followed by unloading and reloading in compression.

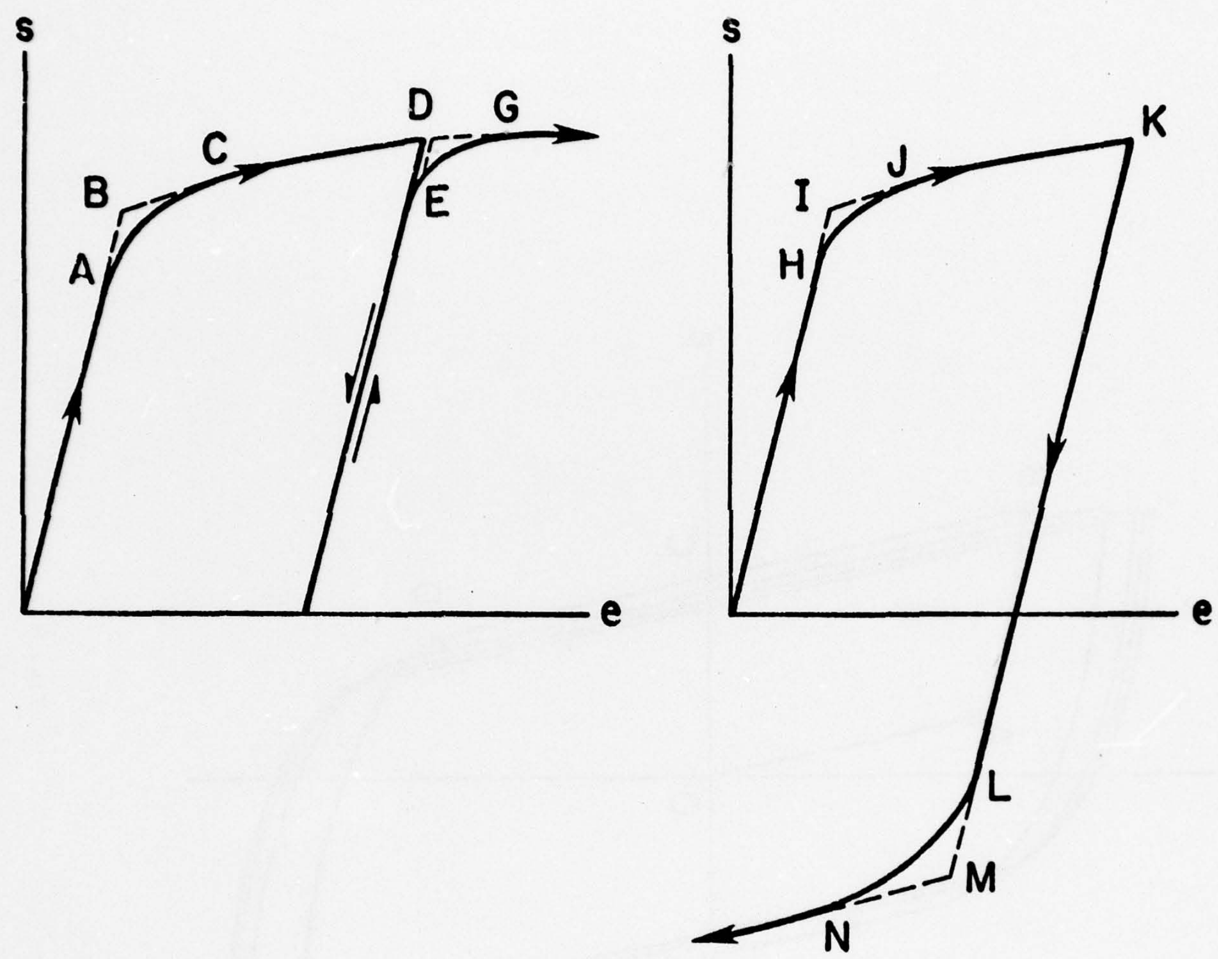
Fig. 2 Mechanical response of a typical ductile metal in cyclic loading, exhibiting saturation hardening and plotted in the stress-strain (s-e) plane.

Fig. 3(a) Experimental stress-strain curves for 304 stainless steel in cyclic tension-compression, as reported in Ref. [13].

Fig. 3(b) Comparison of theoretical cyclic stress-strain behavior for 304 stainless steel with the experimental data from Ref. [13]. The theoretical stress-strain curves (——) are calculated using the constitutive coefficients (90); and comparison with the experimental data (----) is shown for the first two strain cycles only, since curves for additional cycles would crowd the figure [compare with Fig. 3(a)].

Fig. 4 Comparison of the theoretically determined cyclic stress-strain curve (——) for 2024-T351 aluminum alloy and corresponding experimental data (----) from the first loading program (initially between +0.02 and -0.01 strain).

Fig. 5 Comparison of the theoretically determined cyclic stress-strain curve (——) for 2024-T351 aluminum alloy and corresponding experimental data (----) from the second loading program (initially between +0.01 and -0.02 strain).



(a)

(b)

Fig. 1

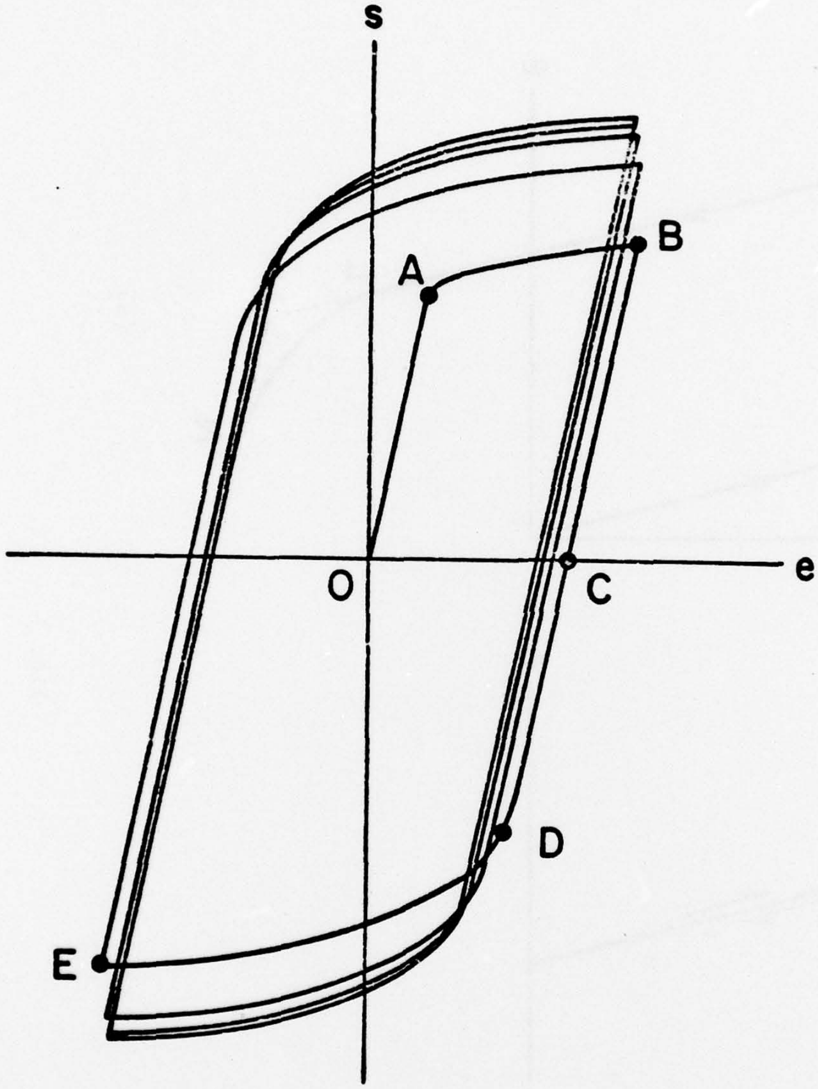


Fig. 2

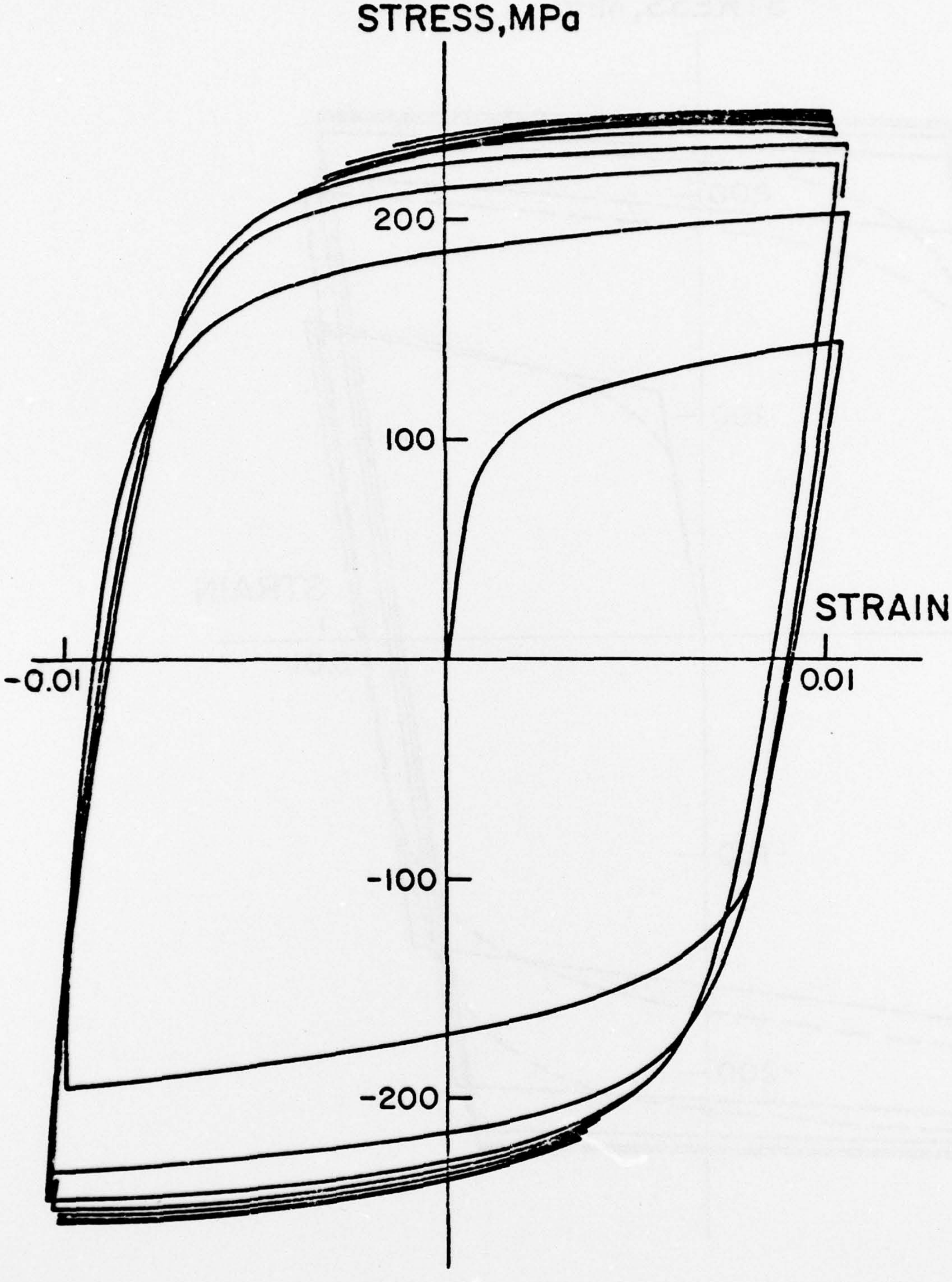


Fig. 3(a)

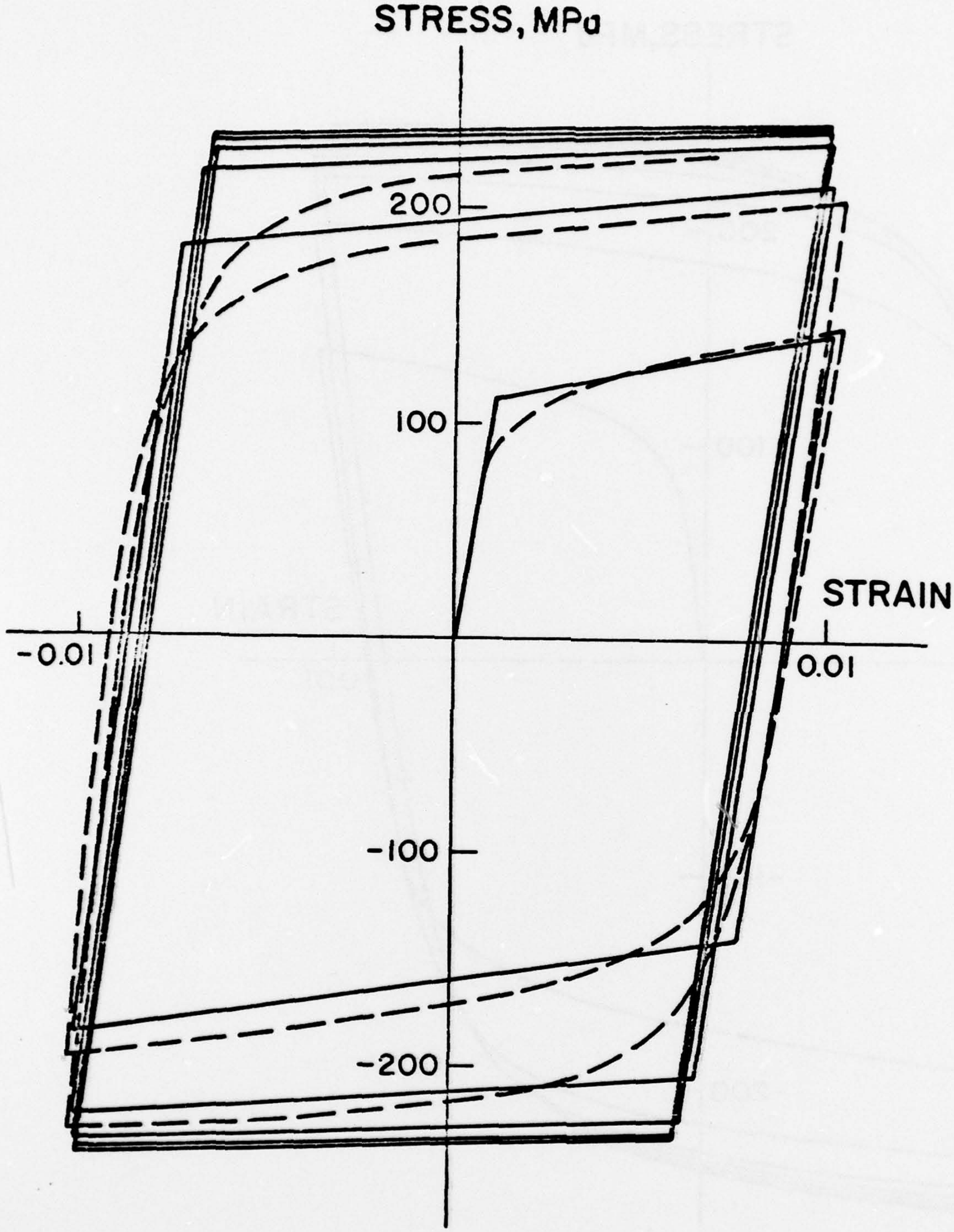


Fig. 3(b)

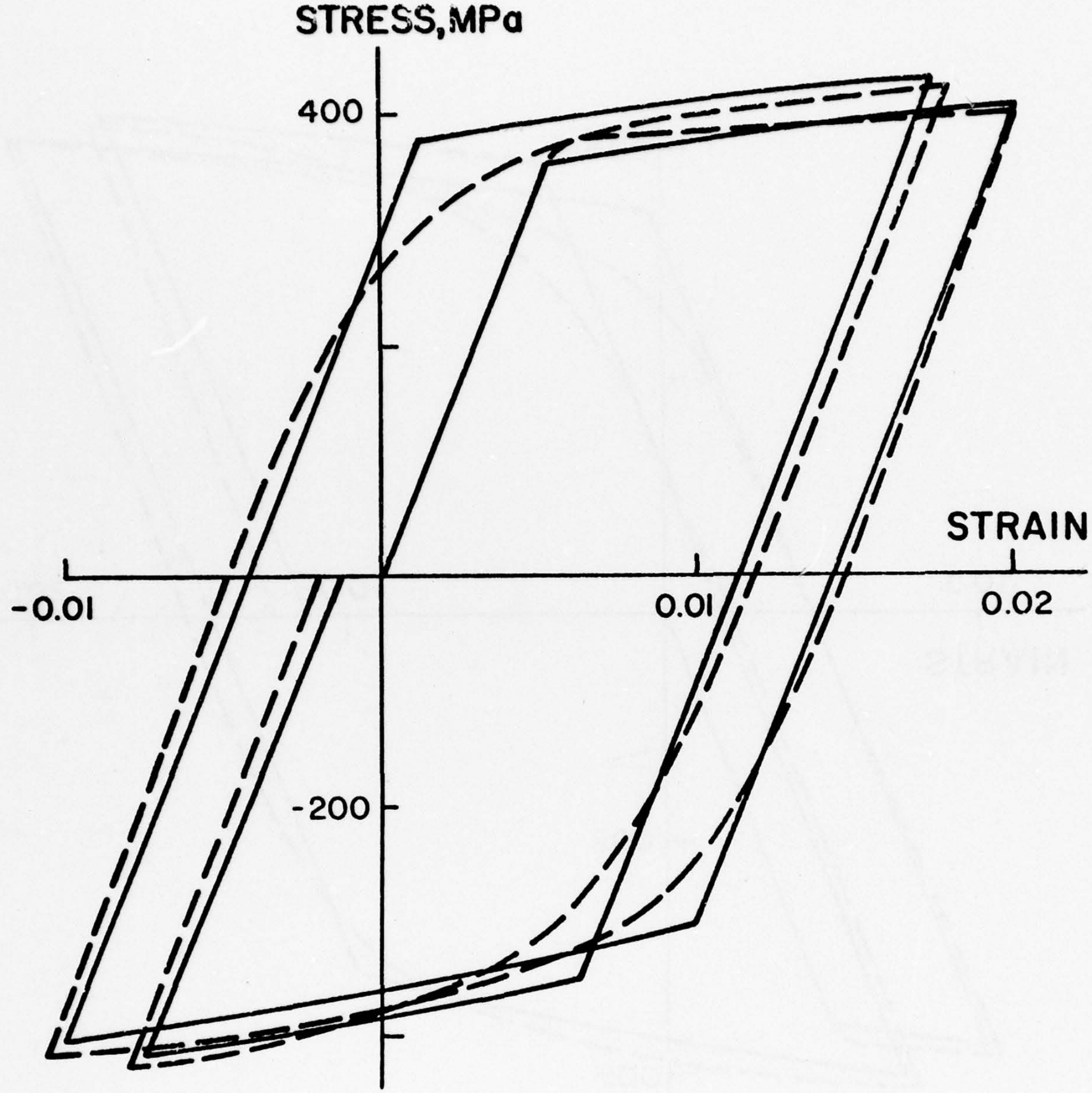


Fig. 4

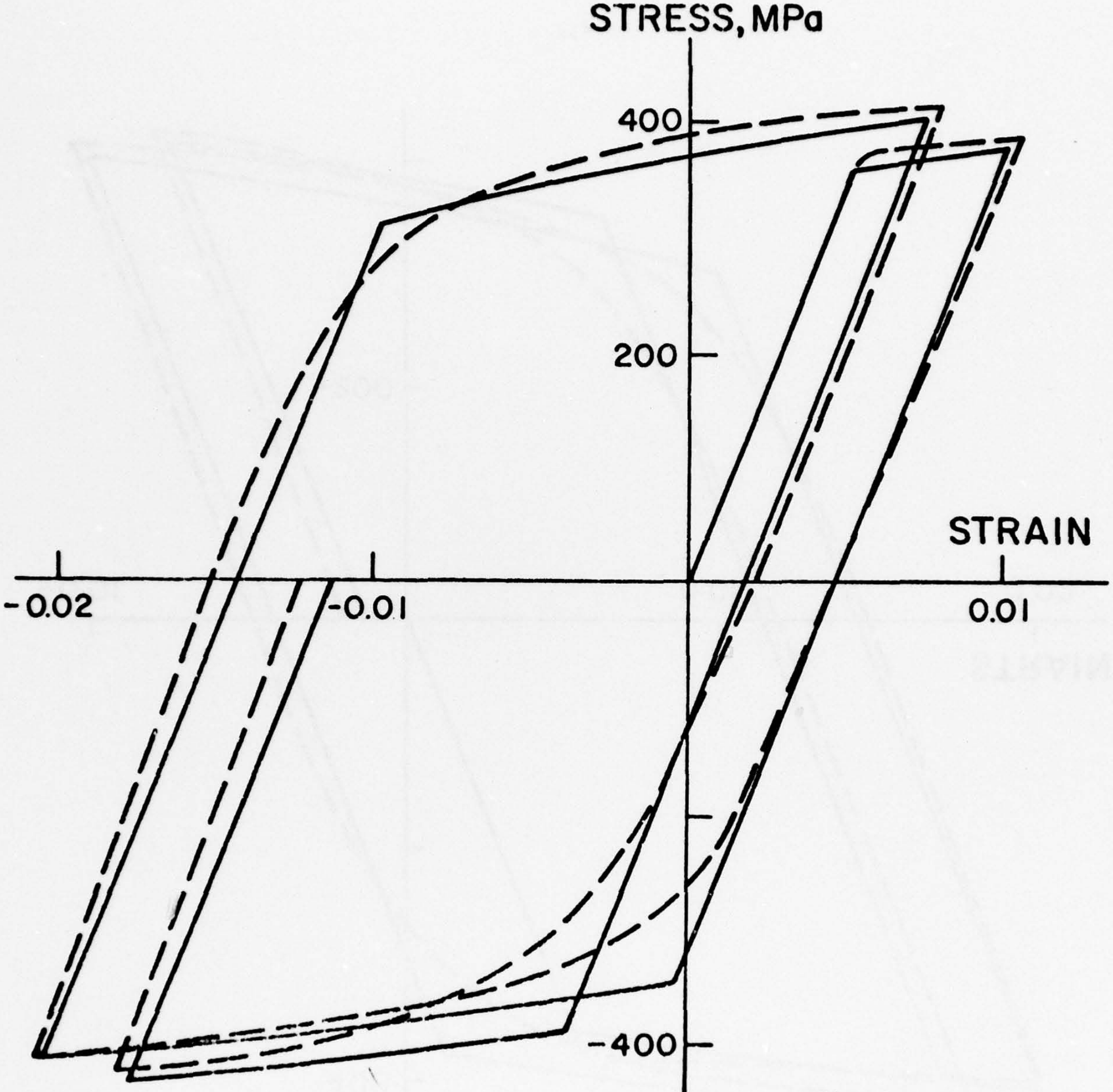


Fig. 5

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This paper is concerned with a special class of hardening response functions for small deformation of elastic-plastic materials, its application to isotropic metals, and comparison of the theoretical results with experimental cyclic stress-strain curves for two different metals. The theoretical development is carried out within the scope of the existing purely mechanical theory for the "rate-independent" response of elastic-plastic materials, which admits
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20. Abstract (continued)

the existence of a single loading function, as well as certain accepted idealizations. After summarizing the basic equations for small deformation, detailed attention is given to the development of a special form of the hardening response function, motivated mainly by the observation that the stress-strain curves for uniaxial cyclic loading of a fairly large class of metals attain -- after several cycles -- the so-called saturation hardening. We exploit this property; and, in the case of isotropic metals, systematically derive some restrictions on the constitutive coefficients in the loading function and the hardening response. Comparison of the results with two sets of experimental data, obtained from uniaxial cyclic loading of a 304 stainless steel and a 2024 aluminum alloy, shows good agreement within the understood idealizations of the basic theory.

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